# ON THE THEORY OF UNLOADING WAVES

**(K TEORII VOLNY RAZGRUZKI)**

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A. M. SKOBEEV
(Moscow)

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The problem considered is the propagation of unloading waves, which was proposed by Rakhmatulin [1]; the existence and uniqueness of functions describing unloading waves is shown; certain qualitative properties of these functions are studied. It is shown that as $t \to \infty$, the speed of propagation of unloading waves approaches asymptotically the speed of propagation of elastic distortions.

**1. Formulation of the problem.** One seeks functions $u(x, t)$, $v(x, t)$, $f(x)$ such that the continuously differentiable function $f(x)$ divides the domain $x \geqslant 0$ of the $x$, $t$ plane into two domains $D_1$ and $D_2$; the functions $u(x, t)$ and $v(x, t)$ are continuous on $x \geqslant 0$, continuously differentiable on the domains $D_1$ and $D_2$, and satisfy:

in the domain $D_1$ the system of equations

$$v_x (x, t) = u_t (x, t), \qquad v_t (x, t) = a^2 (u)\, u_x (x, t) \tag{1.1}$$

and the condition

$$u (x, t_1) \geqslant u (x, t_2) \quad \text{for } t_1 > t_2$$

in the domain $D_2$ the system of equations

$$v_x (x, t) = u_t (x, t)$$
$$v_t (x, t) = a_0^2 u_x (x, t) + [a^2 (u) - a_0^2] \frac{du (x, f (x))}{dx} \tag{1.2}$$

and the condition

$$u (x, t_1) < u (x, t_2) \quad \text{for } t_1 > t_2$$

Besides

$$\sigma(u(0, t)) = p(t) \quad \text{for } t \geqslant 0$$
$$\sigma(u(0, t)) = q(t) \quad \text{for } \theta > t \geqslant \tau(0)$$
$$u(x, \tau(0)) = 0, \quad v(x, \tau(0)) = 0 \quad \text{for } x \geqslant 0$$

where $p(t)$ and $q(t)$ are continuously differentiable functions such that

$$p(0) = q(0), \quad p'(t) \leqslant 0, \quad q'(t) \geqslant 0$$

The function $a(u)$ is a continuously differentiable monotone decreasing function, defined by the equation (see Fig. 1)

$$a^2 = \frac{1}{\rho}\frac{d\sigma(\varepsilon)}{d\varepsilon} \qquad (a(u) = a_0 \quad \text{for } u \leqslant \varepsilon^*)$$

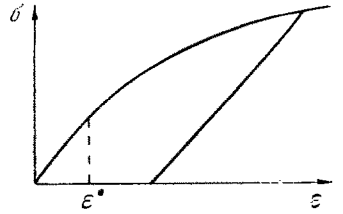

Fig. 1.

where $\rho$ is the density for $u = 0$, that is $\rho$ = const; the function $\tau(\varepsilon)$ is defined by the equation $\sigma(\varepsilon) = q(\tau)$.

2. *Lemma 2.1.* If there exist functions $u(x, t)$, $v(x, t)$, $f(x)$ satisfying the conditions specified above, then

$$\frac{1}{a(\varepsilon(x))} \geqslant \frac{df}{dx} \geqslant \frac{1}{a_0} \tag{2.1}$$

where here as well as in later considerations, we have set

$$\varepsilon(x) = u(x, f(x))$$

*Proof.* Let us suppose that at a certain point the inequality (2.1) does not hold. Then there is an entire interval $[x_0 - \mu, x_0 + \mu]$, with $\mu > 0$, such that the inequality does not hold throughout this interval. As is known

$$\frac{\partial u}{\partial s} = u_x \cos\varphi + u_t \sin\varphi, \qquad \frac{\partial v}{\partial s} = v_x \cos\varphi + v_t \sin\varphi$$

where $\partial u/\partial s$ and $\partial v/\partial s$ are directional derivatives along $f(x)$ and $\tan\varphi = f'(x)$.

From (1.1) and (1.2) we obtain

$$u_t^+ \cos\varphi\left(f'^2 - \frac{1}{a^2}\right) = \frac{\partial u}{\partial s} f' - \frac{\partial v}{\partial s}\frac{1}{a^2} \quad \text{on} \quad D_1 \tag{2.2}$$
$$u_t^- \cos\varphi\frac{a_0^2}{a^2}\left(f'^2 - \frac{1}{a^2}\right) = \frac{\partial u}{\partial s} f' - \frac{\partial v}{\partial s}\frac{1}{a^2} \quad \text{on} \quad D_2 \tag{2.3}$$

where

$$u_t^+ = \lim u_t(x, t) \quad \text{for } (x, t) \in D_1, \qquad (x, t) \to (x, f(x))$$
$$u_t^- = \lim u_t(x, t) \quad \text{for } (x, t) \in D_2, \qquad (x, t) \to (x, f(x))$$

Since $u_t^+ \geqslant 0$, $u_t^- \leqslant 0$, and (2.1) does not hold, from (2.2) and (2.3) it follows that

$$\frac{\partial u}{\partial s} f' - \frac{\partial v}{\partial s}\frac{1}{a^2} = 0 \tag{2.4}$$

Further, notice that $f(0) = 0$ and $f(x) \geqslant x/a_0 + \tau(\varepsilon^*)$, which follows from the fact that, in the domain bounded by the straight lines $t = \tau(0)$ and $t = x/a_0 + \tau(\varepsilon^*)$, the systems (1.1) and (1.2) coincide. Let us suppose that there exists a point $x_1$ such that $f'(x_1) < 0$; then $f(x)$ would have a minimum on the interval $(x, \infty)$, say at the point $x_2$, and there would exist an interval $[x_2 - \mu_1,\ x_2 + \mu_1]$, with $\mu_1 > 0$, on which

$$\frac{1}{a_0} > f' > -\frac{1}{a_0} \tag{2.5}$$

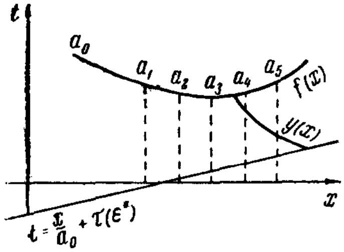


Fig. 2.

that is, (2.1) does not hold. As has already been shown, on this interval (2.4) holds. Consider the interval $[x_2 - \mu_2,\ x_2 + \mu_2]$, $\mu_2 = 1/2\ \mu_1$. Let us set

$$u_1(x, t) = \begin{cases} u(x, t) & \text{for } t \leqslant f(x) \\ \varepsilon(x) & \text{for } t > f(x) \end{cases}$$

Then, for arbitrary $x_3$ in $[x_2 - \mu_2,\ x_2 + \mu_2]$, there exists a function $t = y(x)$ defined for all $x \geqslant x_3$ and satisfying the equation

$$\frac{dy(x)}{dx} = -\frac{1}{a(u_1(x, y))} \tag{2.6}$$

for which $y(x_3) = f(x_3)$, see Fig. 2. In Fig. 2 one has a graph of the function $t = y(x)$, and also of the straight line $t = x/a_0 + \tau(\varepsilon^*)$, and the coordinates of the points $a_0, a_1, \ldots, a_5$ are the following

$$a_0 = \{x_1,\ f(x_1)\}, \quad a_1 = \{x_2 - \mu_1,\ f(x_2 - \mu_1)\}, \qquad a_5 = \{x_2 + \mu_1,\ f(x_2 + \mu_3)\}$$
$$a_2 = \{x_2 - \mu_2,\ f(x_2 - \mu_2)\}, \quad a_3 = \{x_2,\ f(x_2)\},$$

Let us prove that

$$y(x) < f(x) \qquad \text{for } x > x_3$$

Indeed, if $x > x_3$ and $x$ is in the interval $[x_2 - \mu_1,\ x_2 + \mu_1]$, then

$$\frac{dy}{dx} \leqslant -\frac{1}{a_0} < \frac{df}{dx}, \qquad \text{or} \qquad y(x) < f(x)$$

Further, from (2.5) and (2.6) we have

$$f(x_3) - f(x_2) \leqslant \frac{|x_3 - x_2|}{a_0} \leqslant \frac{\mu_2}{a_0}, \qquad y(x_3) - y(x_2 + \mu_1) \geqslant \frac{x_2 - x_3 + \mu_1}{a_0} \geqslant \frac{\mu_2}{a_0} \tag{2.7}$$

Since $f(x_3) = y(x_3)$, it follows that $f(x_2) - y(x_2 + \mu_1) > 0$; and for $x > x_2 + \mu_1$ we have $y(x) < y(x_2 + \mu_1)$. From this, since $f(x) \geqslant f(x_2)$, one deduces that $y(x) < f(x)$ for all $x > x_3$.

Now, since $y(x) < f(x)$ for $x > x_3$, it follows that $(x, y(x))$ is contained in $D_1$ and that the curve is a characteristic of the system (1.1).

But $y' < 0$, and thus the curve $t = y(x)$ must intersect the straight line $t = x/a_0 + \tau(\varepsilon^*)$. Along this straight line one has $u = \text{const}$, $v = \text{const}$, and along $t = y(x)$ one has

$$v + \psi(u) = \text{const} \left(\psi(u) = \int_0^u a(\xi)\, d\xi\right) \tag{2.8}$$

thus

$$v(x, f(x)) + \psi(\varepsilon(x)) = \text{const} \quad \text{for } x \in [x_2 - \mu_2, x_2 + \mu_2] \tag{2.9}$$

Thus, for $x$ in $[x_2 - \mu_2, x_2 + \mu_2]$

$$\frac{\partial v}{\partial s} + a\frac{\partial u}{\partial s} = 0 \tag{2.10}$$

and this, together with (2.4), gives $\partial u/\partial s = 0$, $\partial v/\partial s = 0$, or

$$\varepsilon(x) = \text{const}, \ v(x, f(x)) = \text{const} \qquad (x \in [x_2 - \mu_2, x_2 + \mu_2])$$

This in turn implies that, in the domain bounded by $t = f(x)$ and the characteristics of the system

$$t + \frac{x}{a_0} - f(x_2 - \mu_2) - \frac{x_2 - \mu_2}{a_0} = 0, \qquad t - \frac{x}{a_0} - f(x_2 + \mu_2) + \frac{x_2 + \mu_2}{a_0} = 0$$

one has $u = \text{const}$, $v = \text{const}$, which contradicts the fact that this domain lies in $D_2$.

Thus, finally, $f'(x) \geqslant 0$. This implies that for arbitrary $x$ there exists a characteristic of the system (1.1) which passes through the point $(x, f(x))$ and intersects $t = x/a_0 + \tau(\varepsilon^*)$; that is, that (2.8) and (2.10) hold throughout $D_1$. Then, if for a certain $x$ the inequality (2.1) did not hold, there would exist an interval for which (2.4) and (2.10) hold simultaneously, and this is impossible, as has just been shown. Consequently, the required inequality has been proved.

As was shown by Rakhmatulin [1], from the lemma just proved it follows that the problem of determining $f(x)$ reduces to the solution of the following system of functional equations

$$a_0 f(x_1) + x_1 = a_0 t \tag{2.11}$$

$$a_0 f(x_2) - x_2 = a_0 t \qquad (2.12)$$

$$\frac{\sigma(\varepsilon(x_2)) + \sigma(\varepsilon(x_1))}{2} + \rho a_0 \frac{\psi(\varepsilon(x_2)) - \psi(\varepsilon(x_1))}{2} = p(t) \qquad (2.13)$$

$$f(x) - \frac{x}{a(\varepsilon(x))} = \tau(\varepsilon(x)) \qquad (2.14)$$

and the problem of determining $u(x, t)$ and $v(x, t)$ reduces to the usual boundary value problem for the system (1.1) and (1.2).

**3. Discussion of the fundamental system of equations.** Let us establish certain simple properties of the system (2.11) to (2.14), based on the assumption that it possesses a continuous solution.

3.1. Let us prove that $f(x) \leqslant x/a_1$. Here, and in what follows, $a_1 = a(\varepsilon(0))$, $x \geqslant 0$. Indeed, from equation (2.14) we have $f(x) \leqslant x/a(\varepsilon(x))$; the function $\tau(\varepsilon)$ is defined only for $\varepsilon(x) \leqslant \varepsilon(0)$, and hence $a(\varepsilon(x)) \geqslant a_1$; this gives the desired inequality.

3.2. Let us prove that $\varepsilon(x) > \varepsilon^*$. Recall that $a(\varepsilon) = a_0$ for $\varepsilon \leqslant \varepsilon^*$. Let us suppose that there exists an $x_1$ such that $\varepsilon(x_1) \leqslant \varepsilon^*$. Then equations (2.12) and (2.14) give

$$a_0 f(x_1) - x_1 = a_0 t, \qquad a_0 f(x_1) - x_1 = a_0 \tau(\varepsilon)$$

and since $t \geqslant 0$, and $\tau(\varepsilon) \leqslant \tau(\varepsilon^*) < 0$, this system has no solutions, and the required inequality follows.

3.3. Let us set

$$\Phi(\xi) = \frac{\sigma(\xi)}{2} + \rho a_0 \frac{\psi(\xi)}{2}, \qquad \Psi(\xi) = -\frac{\sigma(\xi)}{2} + \rho a_0 \frac{\psi(\xi)}{2}$$

Then

$$2\Phi'(\xi) = \rho a(a_0 + a), \quad 2\Psi'(\xi) = \rho a(a_0 - a), \quad \text{or } \Phi' > 0, \; \Psi' > 0$$

In particular, there exists a $\delta$ such that

$$\Phi'(\xi_1) \geqslant A > B \geqslant \Psi'(\xi_2) \quad \text{for } \xi_1, \xi_2 \in [\varepsilon(0) - \delta, \varepsilon(0)] \qquad (3.1)$$

Let us put, in the system (2.11) to (2.14)

$$p(t) = p_i(t), \qquad \tau(\varepsilon) = \tau_i(\varepsilon)$$

The solution functions of this system will be denoted in the sequel by $f_i(x)$ and $\varepsilon_i(x)$.

*Lemma 3.1.* Given two systems of the form (2.11) to (2.14) such that

$$p_1(0) = p_2(0), \qquad p_1(t) \geqslant p_2(t), \qquad \tau_1 \ \varepsilon) \leqslant \tau_2(\varepsilon)$$

Then

$$\varepsilon_1(x_1) \geqslant \varepsilon_2(x_2) \quad \text{for } x_1 < x_2 \tag{3.2}$$

*Proof*. Let us suppose that the lemma is not valid, i.e. that there exist $x_1 < x_2$ such that $\varepsilon_1(x_1) < \varepsilon_2(x_2)$. Consider two sequences

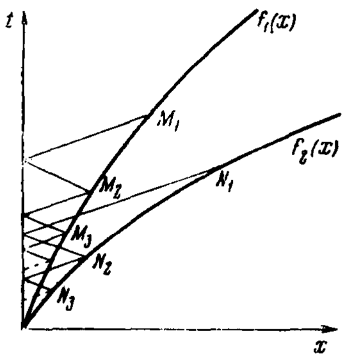


Fig. 3.

$$x_1 = x_1^1, \quad x_2^1, \quad x_3^1, \ldots, \quad x_2 = x_1^2, \quad x_2^2, \ x_3^2, \ldots$$

which satisfy

$$t_i^j \equiv f_i^j - \frac{x_i^j}{a_0} = f_{i+1}{}^j + \frac{x_{i+1}{}^j}{a_0} \tag{3.3}$$

such sequences will be called sequences of type I.

These sequences are shown in Fig. 3, where

$$N_i = \{x_i^2, \ f_2(x_i^2)\}, \qquad N_i = \{x_i^1, \ f_1(x_i^1)\}$$

are the abscissas of the points of sequences of type I, starting with $x_2$ and $x_1$, and $x_i{}^j$ and $x_{i+1}{}^j$ are given by (3.3).

Here we have employed the notation

$$f_i^j = f_j(x_i^j), \qquad \varepsilon_i^j = \varepsilon_j(x_i^j), \qquad \Delta_i = \varepsilon_i^2 - \varepsilon_i^1, \qquad a_i^j = a(\varepsilon_i^j), \qquad \tau_i^j = \tau_j(\varepsilon_i^j)$$

It is obvious that for each $x_i{}^j$ one may obtain from (3.3) the corresponding $x_{i+1}{}^j$ in such a way that, for arbitrary $x_1$ and $x_2$ there exist sequences of type I, beginning with $x_1$ and $x_2$. Further

$$\frac{t_{i+1}{}^j}{t_i^j} = \frac{a_0 f_{i+1}{}^j - x_{i+1}{}^j}{a_0 f_{i+1}{}^j + x_{i+1}{}^j} = \frac{a_0\xi - 1}{a_0\xi + 1}, \qquad \xi = \frac{f_{i+1}{}^j}{x_{i+1}{}^j} \leqslant \frac{1}{a_1}$$

as follows from property 3.1; thus

$$\frac{t_{i+1}{}^j}{t_i^j} \leqslant \frac{a_0 - a_1}{a_0 + a_1} \tag{3.4}$$

This implies that $t_i{}^j \to 0$ as $i \to \infty$ and that $x_i{}^j \to 0$ as $i \to \infty$; because, from equation (2.11) one has $x_i{}^j \leqslant a_0 t_i{}^j$. Let us suppose that

$$\varepsilon_i^2 > \varepsilon_i^1, \qquad t_i^2 \geqslant t_i^1$$

then

$$\Phi(\varepsilon_i^2) - \Psi(\varepsilon_{i+1}^2) = p_2(t_i^2) \qquad \Phi(\varepsilon_i^1) - \Psi(\varepsilon_{i+1}^1) = p_1(t_i^1) \qquad p_2(t_i^2) \leqslant p_1(t_i^1)$$

From this, it follows that

$$\Psi(\varepsilon_{i+1}^2) - \Psi(\varepsilon_{i+1}^1) \geqslant \Phi(\varepsilon_i^2) - \Phi(\varepsilon_i^1)$$

or

$$\Delta_{i+1} \geqslant K_i \Delta_i, \qquad K_i = \frac{a(\xi_1)[a_0 + a(\xi_1)]}{a(\xi_2)[a_0 - a(\xi_2)]} \qquad \begin{pmatrix} \xi_1 \in [\varepsilon_i^2, \varepsilon_i^1] \\ \xi_2 \in [\varepsilon_{i+1}^2, \varepsilon_{i+1}^1] \end{pmatrix}$$

Let us prove that $\varepsilon_i^2 > \varepsilon_i^1$, $t_i^2 \geqslant t_i^1$ imply that $t_{i+1}^2 \geqslant t_{i+1}^1$. From equation (2.14) and the relation (3.3) one obtains

$$t_{i+1}^j = \frac{(a_0 - a_{i+1}^j)\, t_i^j + 2a_{i+1}^j \tau_{i+1}^j}{a_0 + a_{i+1}^j}$$

$$t_{i+1}^2 - t_{i+1}^1 = \frac{(a_0 - a_{i+1}^2)(a_0 + a_{i+1}^1)\, t_i^2 + 2a_{i+1}^2 (a_0 + a_{i+1}^1)\, \tau_{i+1}^2}{(a_0 + a_{i+1}^1)(a_0 + a_{i+1}^2)} -$$

$$- \frac{(a_0 - a_{i+1}^1)(a_0 + a_{i+1}^2)\, t_i^1 + 2a_{i+1}^1 (a_0 + a_{i+1}^2)\, \tau_{i+1}^1}{(a_0 + a_{i+1}^1)(a_0 + a_{i+1}^2)} =$$

$$= \frac{a_0 (a_{i+1}^1 - a_{i+1}^2)(t_i^1 + t_i^2 - 2\tau_{i+1}^2) + (a_0^2 - a_{i+1}^1 a_{i+1}^2)(t_i^2 - t_i^1)}{(a_0 + a_{i+1}^1)(a_0 + a_{i+1}^2)} +$$

$$+ \frac{2a_{i+1}^2 (a_0 + a_{i+1}^1)(\tau_{i+1}^2 - \tau_{i+1}^1)}{(a_0 + a_{i+1}^1)(a_0 + a_{i+1}^2)} \geqslant 0$$

And, since it was already shown that $\Delta_{i+1} > 0$, one obtains

$$a_{i+1}^1 \geqslant a_{i+1}^2, \qquad \tau_{i+1}^2 \geqslant \tau_{i+1}^1, \qquad t_{i+1}^2 \geqslant t_{i+1}^1$$

In particular, therefore

$$t_1^2 - t_1^1 = \left(\frac{1}{a_1^2} - \frac{1}{a_0}\right) x_2 - \left(\frac{1}{a_1^1} - \frac{1}{a_0}\right) x_1 - (\tau_1^2 - \tau_1^1) \geqslant 0$$

Since $\Delta_1 > 0$, $t_1^2 \geqslant t_1^1$, then, for every $i$ one has $t_i^2 \geqslant t_i^1$, $\Delta_{i+1} \geqslant K_i \Delta_i$, from which

$$\Delta_{i+1} \geqslant \Big(\prod_{n=1}^{i} K_n\Big)\, [\varepsilon_2(x_2) - \varepsilon_1(x_1)] \tag{3.5}$$

Employing $x_i^j \to 0$ as $i \to \infty$ and the continuity of the $\varepsilon_j(x)$, one obtains that all the $x_i^j$, save possibly a finite number of them, lie in the neighborhood $[0, \delta_1]$, in which

$$\varepsilon_1(x) \in [\varepsilon(0) - \delta, \varepsilon(0)], \qquad \varepsilon_2(x) \in [\varepsilon(0) - \delta, \varepsilon(0)]$$

for $x$ in $[0, \delta_1]$, where $\delta$ is the same as in property 3.3. Then there are just a finite number of the $K_n$ which are less that $A/B$, where $A$ and $B$ are the same as in property 3.3; that is, from (3.5) we obtain $\Delta_i \to \infty$ as $i \to \infty$, which contradicts the continuity of $\varepsilon_j(x)$ at $x = 0$.

*Theorem 3.1.* The solution of the system (2.11) to (2.14), if it exists, is unique in the class of continuous functions.

Indeed, let us suppose that the system (2.11) to (2.14) has two solutions $\varepsilon_1(x)$, $f_1(x)$ and $\varepsilon_2(x)$, $f_2(x)$. It is clear that $\varepsilon_1(x)$ and $f_1(x)$ satisfy the system (2.11) to (2.14) with $p_1(t) = p(t)$, $\tau_1(\varepsilon) = \tau(\varepsilon)$, and the functions $\varepsilon_2(x)$ and $f_2(x)$ satisfy the same system, but with $p_2(t) = p(t)$, and $\tau_2(\varepsilon) = \tau(\varepsilon)$. From the fundamental Lemma 3.1 we have $\varepsilon_1(x) \geqslant \varepsilon_2(x)$ and $\varepsilon_2(x) \geqslant \varepsilon_1(x)$; hence $\varepsilon_1(x) = \varepsilon_2(x)$, and from (2.14) we obtain $f_1(x) = f_2(x)$. This is what was to be proved.

*Lemma 3.3.* If the system (2.11) to (2.14) has a solution $\varepsilon(x)$, $f(x)$, then $\varepsilon(x)$ is a monotone decreasing function.

The proof follows immediately from Lemma 3.1 and Theorem 3.1.

*Lemma 3.3.* If the system (2.11) to (2.14) has a solution in the class of continuously differentiable functions, then $f'(x)$ satisfies the inequality (2.1).

*Proof.* Differentiating equation (2.14) we obtain

$$\frac{df}{dx} = \frac{1}{a(\varepsilon(x))} + \left[x \frac{d}{d\varepsilon}\frac{1}{a(\varepsilon)} + \frac{d\tau}{d\varepsilon}\right]\frac{d\varepsilon}{dx}$$

and, because $\varepsilon'(x) \leqslant 0$, $a'(\varepsilon) \leqslant 0$, $\tau'(\varepsilon) \geqslant 0$, we obtain $f'(x) \leqslant a^{-1}(\varepsilon(x))$. Further, obviously, the inequality $f' \geqslant a_0^{-1}$ is equivalent to the fact that equation (2.12) has, for all $t > 0$, not more than one solution.

Let us suppose that, for a certain $t_0$, equation (2.12) did possess solutions $x_2^1$ and $x_2^2$, with $x_2^1 \neq x_2^2$. Let us choose a solution of equation (2.11) and call it $x_1$. Then, from equation (2.13) we have that $\varepsilon(x_2^1) = \varepsilon(x_2^2) > \varepsilon^*$, and from equations (2.12) and (2.14) we obtain that $x_2^1 = x_2^2$, which contradicts $x_2^1 \neq x_2^2$; therefore $f'(x) > a_0^{-1}$ and the lemma is proved.

Let us note that from Lemma 3.3 it follows that, if the system (2.11) to (2.14) has a solution, the original problem also has a solution.

*Lemma 3.4.* Suppose that one has two systems of the form (2.11) to (2.14), with solutions $\varepsilon_1(x)$, $f_1(x)$, and $\varepsilon_2(x)$, $f_2(x)$. Then

$$|\varepsilon_1(x) - \varepsilon_2(x)| \leqslant (\rho a_1^2)^{-1}\mu$$

$$a = \{\min a(\varepsilon_j(0))\} \quad (j = 1, 2), \qquad \lambda = \max|\tau_1(\varepsilon) - \tau_2(\varepsilon)$$

$$\mu = \max|p_1(t) - p_2(t)| + \max|p_1(t+2\lambda) - p_1(t)|$$

*Proof*. Let us consider the difference $\varepsilon_1(x) - \varepsilon_2(x)$ at a certain point $x$, and suppose that $\varepsilon_2(x) - \varepsilon_1(x) < 0$; the other alternative can be handled merely by permuting the subscripts 1 and 2. Then

$$t_1 - t_2 = x\left[\frac{1}{a(\varepsilon_1(x))} - \frac{1}{a(\varepsilon_2(x))}\right] + \tau_1(\varepsilon_1(x)) - \tau_2(\varepsilon_2(x)) =$$
$$= x\frac{a(\varepsilon_2(x)) - a(\varepsilon_1(x))}{a(\varepsilon_2(x))\,a(\varepsilon_1(x))} + \tau_1(\varepsilon_1(x)) - \tau_1(\varepsilon_2(x)) +$$
$$+ \tau_1(\varepsilon_2(x)) - \tau_2(\varepsilon_2(x)) \geqslant -|\tau_1(\varepsilon_2(x)) - \tau_2(\varepsilon_2(x))| \geqslant -\lambda$$

From this one obtains $t_2 - t_1 \leqslant 2\lambda$. Consider two sequences of type I

$$x = x_1^1, \quad x_2^1, \quad x_3^1, \ldots, \quad x = x_1^2, \quad x_2^2, \quad x_3^2, \ldots$$

from inequality (2.1) it follows that

$$x_i^j > x_{i+1}^j, \qquad \varepsilon_{i+1}^j \geqslant \varepsilon_i^j \qquad (j = 1, 2)$$

Here and in what follows we shall employ the notation of Lemma 3.1.

Let

$$\varepsilon_i^1 \geqslant \varepsilon_i^2 \quad \text{for } i \leqslant n, \qquad t_i^2 - t_i^1 \leqslant 2\lambda \quad \text{for } i \leqslant n-1$$

Then

$$t_n^2 - t_n^1 = \frac{(a_0 - a_n^2)\,t_{n-1}^2 + 2a_n^2\tau_n^2}{a_0 + a_n^2} - \frac{(a_0 - a_n^1)\,t_{n-1}^1 + 2a_n^1\tau_n^1}{a_0 + a_n^1} =$$
$$= \frac{a_0(a_n^1 - a_n^2)(t_{n-1}^1 + t_{n-1}^2 - 2\tau_n^1) + (a_0^2 - a_n^1 a_n^2)(t_{n-1}^1 - t_{n-1}^2)}{(a_0 + a_n^1)(a_0 + a_n^2)} +$$
$$+ \frac{2a_n^1(a_0 + a_n^2)(\tau_n^1 - \tau_n^2)}{(a_0 + a_n^1)(a_0 + a_n^2)} \leqslant \frac{2(a_0^2 - a_n^1 a_n^2) + 2a_n^1(a_0 + a_n^2)}{(a_0 + a_n^1)(a_0 + a_n^2)}\lambda =$$
$$= \frac{2a_0}{a_0 + a_n^2}\lambda \leqslant 2\lambda \quad \text{or} \quad t_n^2 - t_n^1 \leqslant 2\lambda$$

Further

$$p_2(t_i^2) - p_1(t_i^1) = p_2(t_i^2) - p_2(t_i^1 + 2\lambda) + p_2(t_i^1 + 2\lambda) -$$
$$- p_1(t_i^1 + 2\lambda) + p_1(t_i^1 + 2\lambda) - p_1(t_i^1) \geqslant -\mu$$

For $i \leqslant n$ we have

$$\Psi(\varepsilon_i^1) - \Psi(\varepsilon_i^2) \geqslant \Phi(\varepsilon_{i-1}^1) - \Phi(\varepsilon_{i-1}^2) - \mu \quad (\varepsilon_i^1 \geqslant \varepsilon_i^2,\ \varepsilon_i^2 \geqslant \varepsilon_{i-1}^2,\ \varepsilon_i^1 \geqslant \varepsilon_{i-1}^1) \tag{3.6}$$

Therefore, putting $\Psi'(\xi) = g(\xi)$, $\Phi'(\xi) = q(\xi)$, we get

$$\int_{\varepsilon_i^2}^{\varepsilon_i^1} g(x)\,dx - \int_{\varepsilon_{i-1}^2}^{\varepsilon_{i-1}^1} q(x)\,dx + \mu \geqslant 0 \qquad (3.7)$$

that is, setting

$$x(t) = \frac{\varepsilon_{i-1}^1 - \varepsilon_{i-1}^2}{\varepsilon_i^1 - \varepsilon_i^2}\, t + \frac{\varepsilon_{i-1}^1 \varepsilon_i^2 - \varepsilon_{i-1}^2 \varepsilon_i^1}{\varepsilon_i^1 - \varepsilon_i^2}$$

it follows from (3.7), using the mean value theorem, that

$$-\Delta_i - \frac{q(x(\xi))}{g(\xi)}\left[-\Delta_{i-1} - \frac{\mu}{q(x(\xi))}\right] \geqslant 0$$

Further, $x(\xi) \leqslant \xi$, and since $x(\xi)$ is linear

$$x(\varepsilon_i^2) = \varepsilon_{i-1}^2 \leqslant \varepsilon_i^2, \qquad x(\varepsilon_i^1) = \varepsilon_{i-1}^1 \leqslant \varepsilon_i^1, \quad \xi \in [\varepsilon_i^2, \varepsilon_i^1]$$

From this

$$\frac{q(x(\xi))}{g(x)} \geqslant \frac{q(\xi)}{g(\xi)} = \frac{a_0 + a(\xi)}{a_0 - a(\xi)} \geqslant \frac{a_0 + a_1}{a_0 - a_1} = \alpha$$

and, because $-\Delta_i \geqslant 0$, one obtains

$$-\Delta_i \geqslant \alpha\left[-\Delta_{i-1} - \frac{2\mu}{\rho a_1 (a_0 + a_1)}\right] \qquad (3.8)$$

that is

$$-\Delta_i \geqslant \alpha^{i-1}\left[(\varepsilon_1^1 - \varepsilon_1^2) - \sum_{k=0}^{i-2} \frac{1}{\alpha^k} \frac{2\mu}{\rho a_1 (a_0 + a_1)}\right] \qquad (3.9)$$

There are two possible cases: (1) all $-\Delta_n > 0$, and (2) there is $n_1$ such that $-\Delta_{n_1} > 0$, $-\Delta_{n_1+1} \leqslant 0$. In the first case, since the quantities $\Delta_n$ are bounded, from (3.9) we have

$$\varepsilon_1^1 - \varepsilon_1^2 \leqslant \sum_{k=0}^{\infty} \frac{1}{\alpha^k} \frac{2\mu}{\rho a_1 (a_0 + a_1)} \qquad (3.10)$$

while in the second case, from (3.8)

$$\varepsilon_{n_1}^1 - \varepsilon_{n_1}^2 \leqslant \frac{2\mu}{\rho a_1 (a_0 + a_1)}, \quad \text{or} \quad \varepsilon_1^1 - \varepsilon_1^2 \leqslant \sum_{k=0}^{\infty} \frac{1}{\alpha^k} \frac{2\mu}{\rho a_1 (a_0 + \alpha_1)}$$

The conclusion of the lemma follows immediately from this.

*Lemma 3.5*. Let us suppose that for $x$ in $[0, m]$ there exist continuously differentiable functions $f_0(x)$ and $\varepsilon_0(x)$, satisfying the system

(2.11) to (2.14). Then there exist continuously differentiable functions $\varepsilon(x)$ and $f(x)$, defined for $x \geqslant 0$ which satisfy the system (2.11) to (2.14) for $x \geqslant 0$, and also satisfy

$$f(x) = f_0(x), \qquad \varepsilon(x) = \varepsilon_0(x) \quad \text{for } x \in [0, m]$$

*Proof*. Let us define the function $h(x)$ by means of the equation

$$\Phi(h(x)) = \Psi(\varepsilon_0(x)) + p(f_0(x) + a_0^{-1}x)$$

and suppose that $h(m) > \varepsilon^*$. Let us introduce a function $x_1(x)$ defined by the equation

$$x = a(h(x_1)) \frac{a_0 f(x_1) + x_1 - a_0 \tau(\varepsilon(x_1))}{a_0 - a(h x_1)} \quad \text{for } x_1 \leqslant m \tag{3.11}$$

From $h(m) > \varepsilon^*$ and the monotonicity of the functions appearing in (3.11) it is obvious that this equation has a unique solution. Put $f_1(x) = a^{-1}(\varepsilon_1(x))x + \tau(\varepsilon_1(x))$ and $\varepsilon_1(x) = h(x_1(x))$, it is immediately verified that these functions satisfy the system (2.11) to (2.14) on the interval $[0, m_1]$, with $m = x_1(m_1)$; the continuous differentiability of this solution follows immediately from the continuity of $\varepsilon_0'(x)$ and $f_0'(x)$. Let us construct, in an analogous manner, the intervals $[0, m_2]$, $[0, m_3]$,..., and suppose that the process be continued indefinitely. Then we obtain, from (2.11) and (2.12)

$$t_m \geqslant \frac{a_0 + a_1}{a_0 - a_1} t_{m-1} \qquad \left(t_n = f_n(m_n) - \frac{m_n}{a_0}\right)$$

Thus

$$t_m \to \infty, \qquad p(t_m) \to 0, \qquad \varepsilon(t_m) \to \varepsilon_0 \geqslant \varepsilon^*$$

On the other hand, equation (2.13) gives $\varepsilon_0 = 0$, and hence there exists $n$ such that $h(m_n) > \varepsilon^*$, $h(m_{n+1}) \leqslant \varepsilon^*$. Therefore there is an $x^*$ for which $h(x^*) = \varepsilon^*$.

From equation (3.11) it follows that $x \to \infty$ as $x_1(x) \to x^* - 0$; thus, the functions $f_{n+1}(x)$ and $\varepsilon_{n+1}(x)$ are defined for all $x \geqslant 0$ and are solutions of the system (2.11) to (2.14); according to Theorem 3.1 one must have $\varepsilon_{n+1}(x) = \varepsilon_0(x)$, $f_{n+1}(x) = f_0(x)$, when $x \in [0, m]$, and the lemma is proved.

*Note*. We have

$$f(x) - \frac{x}{a_0} = f(x_1(x)) + \frac{x_1(x)}{a_0}$$

that is, for $x \to \infty$:

$$f(x) = \frac{x}{a_0} + \left[f(x^*) + \frac{x^*}{a_0}\right] + o(1)$$

which assures the existence of the asymptotes of $f(x)$. Let us note that the existence of the asymptotes of $f(x)$ has the following physical interpretation: the speed of propagation of the unloading waves, as $x \to \infty$, approaches the speed of propagation of elastic vibrations, $a_0$.

*Theorem 3.2.* If $p(t)$, $\tau(\varepsilon)$, $a(\varepsilon)$ are continuously differentiable functions, and $\tau'(\varepsilon) > 0$ for $\varepsilon$ in the interval $[\varepsilon^*, \varepsilon(0)]$, then the solution of the system (2.11) to (2.14) exists, in the class of continuously differentiable functions.

*Proof.* Consider a sequence of continuously differentiable functions $p_i(t)$, converging uniformly, together with their first derivatives, to $p(t)$, and with $p_i(t) = p(0)$ for $t \leqslant \mu_i (\mu_i > 0)$. Then any system (2.11) to (2.14) with $p(t) = p_i(t)$, $\tau(\varepsilon) = \tau_i(\varepsilon)$, has, in view of Lemma 3.5, a solution in the class of continuously differentiable functions, because in an arbitrary neighborhood of zero one may set

$$\sigma(\varepsilon_i(x)) = p(0), \qquad f_i(x) = a_1^{-1}x$$

Now, from Lemma 3.4

$$|\varepsilon_j(x) - \varepsilon_i(x)| \leqslant (\rho a_1^2)^{-1} \max |p_j(t) - p_i(t)|$$

Hence the sequence $\varepsilon_i(x)$ converges to a continuous function $\varepsilon(x)$, which, together with the function

$$f(x) = \frac{x}{a(\varepsilon(x))} + \tau(\varepsilon(x))$$

constitutes a solution of the system (2.11) to (2.14).

In order to prove the continuous differentiability of the solution, one differentiates equation (2.13) and carries out considerations analogous to those employed in the proof of Lemma 3.4, thus obtaining bounds of the type of (3.10), after which the proof does not present any difficulty.

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